CHAPTER

3

RADIATION INTEGRALS AND AUXILIARY POTENTIAL FUNCTIONS

3.1 INTRODUCTION

In the analysis of radiation problems, the usual procedure is to specify the sources and then require the fields radiated by the sources. This is in contrast to the synthesis problem where the radiated fields are specified, and we are required to find the sources.

It is a very common practice in the analysis procedure to introduce auxiliary functions, known as *vector potentials,* which will aid in the solution of the problems. The most common vector potential functions are the **A** (magnetic vector potential) and **F** (electric vector potential). Another pair is the Hertz potentials $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_h$. *Although the electric and magnetic field intensities* (**E** *and* **H**) *represent physically measurable quantities, among most engineers the potentials are strictly mathematical tools.* The introduction of the potentials often simplifies the solution even though it may require determination of additional functions. While it is possible to calculate the **E** and **H** fields directly from the source-current densities **J** and **M**, as shown in Figure 3.1, it is usually much simpler to calculate the auxiliary potential functions first and then determine the **E** and **H**. This two-step procedure is also shown in Figure 3.1.

The one-step procedure, through path 1, relates the **E** and **H** fields to **J** and **M** by integral relations. The two-step procedure, through path 2, relates the **A** and **F** (or $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_h$) potentials to **J** and **M** by integral relations. The **E** and **H** are then determined simply by differentiating **A** and **F** (or $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_h$). Although the two-step procedure requires both integration and differentiation, where path 1 requires only integration, the integrands in the two-step procedure are much simpler.

The most difficult operation in the two-step procedure is the integration to determine **A** and **F** (or $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_h$). Once the vector potentials are known, then **E** and **H** can always be determined because any well-behaved function, no matter how complex, can always be differentiated.

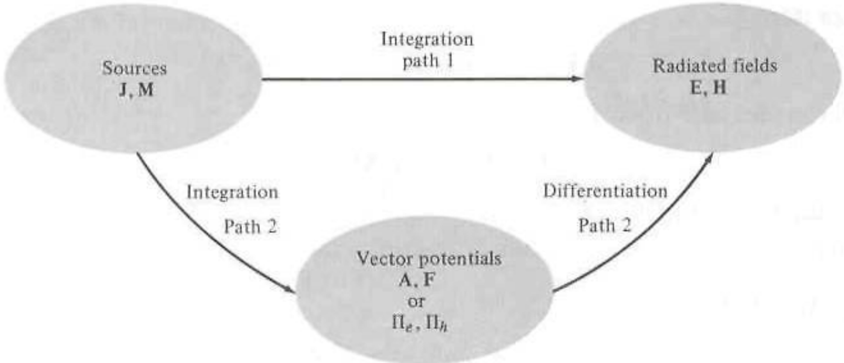

Figure 3.1 Block diagram for computing radiated fields from electric and magnetic sources.

The integration required to determine the potential functions is restricted over the bounds of the sources **J** and **M**. This will result in the **A** and **F** (or $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_h$) to be functions of the observation point coordinates; the differentiation to determine **E** and **H** must be done in terms of the observation point coordinates. The integration in the one-step procedure also requires that its limits be determined by the bounds of the sources.

The vector Hertz potential $\mathbf{\Pi}_e$ is analogous to **A** and $\mathbf{\Pi}_h$ is analogous to **F**. The functional relation between them is a proportionality constant which is a function of the frequency and the constitutive parameters of the medium. In the solution of a problem, only one set, **A** and **F** or $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_h$, is required. The author prefers the use of **A** and **F**, which will be used throughout the book. The derivation of the functional relations between **A** and $\mathbf{\Pi}_e$, and **F** and $\mathbf{\Pi}_h$ are assigned at the end of the chapter as problems. (Problems 3.1 and 3.2).

3.2 THE VECTOR POTENTIAL A FOR AN ELECTRIC CURRENT SOURCE J

The vector potential **A** is useful in solving for the EM field generated by a given harmonic electric current **J**. The magnetic flux **B** is always solenoidal; that is, $\nabla \cdot \mathbf{B} = 0$. Therefore, it can be represented as the curl of another vector because it obeys the vector identity

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \tag{3-1}$$

where **A** is an arbitrary vector. Thus we define

$$\mathbf{B}_A = \mu \mathbf{H}_A = \nabla \times \mathbf{A} \tag{3-2}$$

or

$$\boxed{\mathbf{H}_A = \frac{1}{\mu} \nabla \times \mathbf{A}} \tag{3-2a}$$

where subscript A indicates the field due to the **A** potential. Substituting (3-2a) into Maxwell's curl equation

$$\nabla \times \mathbf{E}_A = -j\omega\mu \mathbf{H}_A \tag{3-3}$$

reduces it to

$$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A = -j\omega \nabla \times \mathbf{A} \tag{3-4}$$

which can also be written as

$$\nabla \times [\mathbf{E}_A + j\omega\mathbf{A}] = 0 \tag{3-5}$$

From the vector identity

$$\nabla \times (-\nabla \phi_e) = 0 \tag{3-6}$$

and (3-5), it follows that

$$\mathbf{E}_A + j\omega\mathbf{A} = -\nabla \phi_e \tag{3-7}$$

or

$$\boxed{\mathbf{E}_A = -\nabla \phi_e - j\omega\mathbf{A}} \tag{3-7a}$$

The scalar function ϕ_e represents an arbitrary electric scalar potential which is a function of position.

Taking the curl of both sides of (3-2) and using the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{3-8}$$

reduces it to

$$\nabla \times (\mu\mathbf{H}_A) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{3-8a}$$

For a homogeneous medium, (3-8a) reduces to

$$\mu\nabla \times \mathbf{H}_A = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{3-9}$$

Equating Maxwell's equation

$$\boxed{\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\epsilon\mathbf{E}_A} \tag{3-10}$$

to (3-9) leads to

$$\mu\mathbf{J} + j\omega\mu\epsilon\mathbf{E}_A = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{3-11}$$

Substituting (3-7a) into (3-11) reduces it to

$$\begin{aligned} \nabla^2\mathbf{A} + k^2\mathbf{A} &= -\mu\mathbf{J} + \nabla(\nabla \cdot \mathbf{A}) + \nabla(j\omega\mu\epsilon\phi_e) \\ &= -\mu\mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\mu\epsilon\phi_e) \end{aligned} \tag{3-12}$$

where $k^2 = \omega^2\mu\epsilon$.

In (3-2), the curl of **A** was defined. Now we are at liberty to define the divergence of **A**, which is independent of its curl. In order to simplify (3-12), let

$$\boxed{\nabla \cdot \mathbf{A} = -j\omega\epsilon\mu\phi_e \Rightarrow \phi_e = -\frac{1}{j\omega\mu\epsilon}\nabla \cdot \mathbf{A}} \tag{3-13}$$

which is known as the *Lorentz condition*. Substituting (3-13) into (3-12) leads to

$$\boxed{\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}} \tag{3-14}$$

In addition, (3-7a) reduces to

$$\mathbf{E}_A = -\nabla\phi_e - j\omega\mathbf{A} = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) \tag{3-15}$$

Once **A** is known, $\mathbf{H}_A$ can be found from (3-2a) and $\mathbf{E}_A$ from (3-15). $\mathbf{E}_A$ can just as easily be found from Maxwell's equation (3-10) with $\mathbf{J} = 0$. It will be shown later how to find **A** in terms of the current density **J**. It will be a solution to the inhomogeneous Helmholtz equation of (3-14).

3.3 THE VECTOR POTENTIAL F FOR A MAGNETIC CURRENT SOURCE M

Although magnetic currents appear to be physically unrealizable, equivalent magnetic currents arise when we use the volume or the surface equivalence theorems. The fields generated by a harmonic magnetic current in a homogeneous region, with $\mathbf{J} = 0$ but $\mathbf{M} \neq 0$, must satisfy $\nabla\cdot\mathbf{D} = 0$. Therefore, $\mathbf{E}_F$ can be expressed as the curl of the vector potential **F** by

$$\mathbf{E}_F = -\frac{1}{\epsilon}\nabla\times\mathbf{F} \tag{3-16}$$

Substituting (3-16) into Maxwell's curl equation

$$\nabla\times\mathbf{H}_F = j\omega\epsilon\mathbf{E}_F \tag{3-17}$$

reduces it to

$$\nabla\times(\mathbf{H}_F + j\omega\mathbf{F}) = 0 \tag{3-18}$$

From the vector identity of (3-6), it follows that

$$\mathbf{H}_F = -\nabla\phi_m - j\omega\mathbf{F} \tag{3-19}$$

where ϕ_m represents an arbitrary magnetic scalar potential which is a function of position. Taking the curl of (3-16)

$$\nabla\times\mathbf{E}_F = -\frac{1}{\epsilon}\nabla\times\nabla\times\mathbf{F} = -\frac{1}{\epsilon}[\nabla\nabla\cdot\mathbf{F} - \nabla^2\mathbf{F}] \tag{3-20}$$

and equating it to Maxwell's equation

$$\nabla\times\mathbf{E}_F = -\mathbf{M} - j\omega\mu\mathbf{H}_F \tag{3-21}$$

leads to

$$\nabla^2\mathbf{F} + j\omega\mu\epsilon\mathbf{H}_F = \nabla\nabla\cdot\mathbf{F} - \epsilon\mathbf{M} \tag{3-22}$$

Substituting (3-19) into (3-22) reduces it to

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M} + \nabla(\nabla\cdot\mathbf{F}) + \nabla(j\omega\mu\epsilon\phi_m) \tag{3-23}$$

By letting

$$\nabla \cdot \mathbf{F} = -j\omega\mu\epsilon\phi_m \Rightarrow \phi_m = -\frac{1}{j\omega\mu\epsilon}\nabla \cdot \mathbf{F} \tag{3-24}$$

reduces (3-23) to

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M} \tag{3-25}$$

and (3-19) to

$$\mathbf{H}_F = -j\omega\mathbf{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F}) \tag{3-26}$$

Once $\mathbf{F}$ is known, $\mathbf{E}_F$ can be found from (3-16) and $\mathbf{H}_F$ from (3-26) or (3-21) with $\mathbf{M} = 0$. It will be shown later how to find $\mathbf{F}$ once $\mathbf{M}$ is known. It will be a solution to the inhomogeneous Helmholtz equation of (3-25).

3.4 ELECTRIC AND MAGNETIC FIELDS FOR ELECTRIC (J) AND MAGNETIC (M) CURRENT SOURCES

In the previous two sections we have developed equations that can be used to find the electric and magnetic fields generated by an electric current source $\mathbf{J}$ and a magnetic current source $\mathbf{M}$. The procedure requires that the auxiliary potential functions $\mathbf{A}$ and $\mathbf{F}$ generated, respectively, by $\mathbf{J}$ and $\mathbf{M}$ are found first. In turn, the corresponding electric and magnetic fields are then determined ($\mathbf{E}_A$, $\mathbf{H}_A$ due to $\mathbf{A}$ and $\mathbf{E}_F$, $\mathbf{H}_F$ due to $\mathbf{F}$). The total fields are then obtained by the superposition of the individual fields due to $\mathbf{A}$ and $\mathbf{F}$ ($\mathbf{J}$ and $\mathbf{M}$).

In summary form, the procedure that can be used to find the fields is as follows:

Summary

1. Specify $\mathbf{J}$ and $\mathbf{M}$ (electric and magnetic current density sources).
2. **a.** Find $\mathbf{A}$ (due to $\mathbf{J}$) using

$$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkR}}{R}dv' \tag{3-27}$$

which is a solution of the inhomogeneous vector wave equation of (3-14).

b. Find $\mathbf{F}$ (due to $\mathbf{M}$) using

$$\mathbf{F} = \frac{\epsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-jkR}}{R}dv' \tag{3-28}$$

which is a solution of the inhomogeneous vector wave equation of (3-25). In (3-27) and (3-28), $k^2 = \omega^2\mu\epsilon$ and R is the distance from any point in the source to the observation point. In a latter section, we will demonstrate that (3-27) is a solution to (3-14) as (3-28) is to (3-25).

3. **a.** Find $\mathbf{H}_A$ using (3-2a) and $\mathbf{E}_A$ using (3-15). $\mathbf{E}_A$ can also be found using Maxwell's equation of (3-10) with $\mathbf{J} = 0$.
   b. Find $\mathbf{E}_F$ using (3-16) and $\mathbf{H}_F$ using (3-26). $\mathbf{H}_F$ can also be found using Maxwell's equation of (3-21) with $\mathbf{M} = 0$.
4. The total fields are then given by

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) - \frac{1}{\epsilon}\nabla\times\mathbf{F} \tag{3-29}$$

or

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H}_A - \frac{1}{\epsilon}\nabla\times\mathbf{F} \tag{3-29a}$$

and

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla\times\mathbf{A} - j\omega\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{F}) \tag{3-30}$$

or

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla\times\mathbf{A} - \frac{1}{j\omega\mu}\nabla\times\mathbf{E}_F \tag{3-30a}$$

Whether (3-15) or (3-10) is used to find $\mathbf{E}_A$ and (3-26) or (3-21) to find $\mathbf{H}_F$ depends largely upon the problem. In many instances one may be more complex than the other or vice versa. In computing fields in the far-zone, it will be easier to use (3-15) for $\mathbf{E}_A$ and (3-26) for $\mathbf{H}_F$ because, as it will be shown, the second term in each expression becomes negligible in that region.

3.5 SOLUTION OF THE INHOMOGENEOUS VECTOR POTENTIAL WAVE EQUATION

In the previous section we indicated that the solution of the inhomogeneous vector wave equation of (3-14) is (3-27).

To derive it, let us assume that a source with current density J_z, which in the limit is an infinitesimal source, is placed at the origin of a x, y, z coordinate system, as shown in Figure 3.2(a). Since the current density is directed along the z-axis (J_z), only an A_z component will exist. Thus we can write (3-14) as

$$\nabla^2 A_z + k^2 A_z = -\mu J_z \tag{3-31}$$

At points removed from the source ($J_z = 0$), the wave equation reduces to

$$\nabla^2 A_z + k^2 A_z = 0 \tag{3-32}$$

Since in the limit the source is a point, it requires that A_z is not a function of direction (θ and ϕ); in a spherical coordinate system, $A_z = A_z(r)$ where r is the radial distance. Thus (3-32) can be written as

$$\nabla^2 A_z(r) + k^2 A_z(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial A_z(r)}{\partial r}\right] + k^2 A_z(r) = 0 \tag{3-33}$$

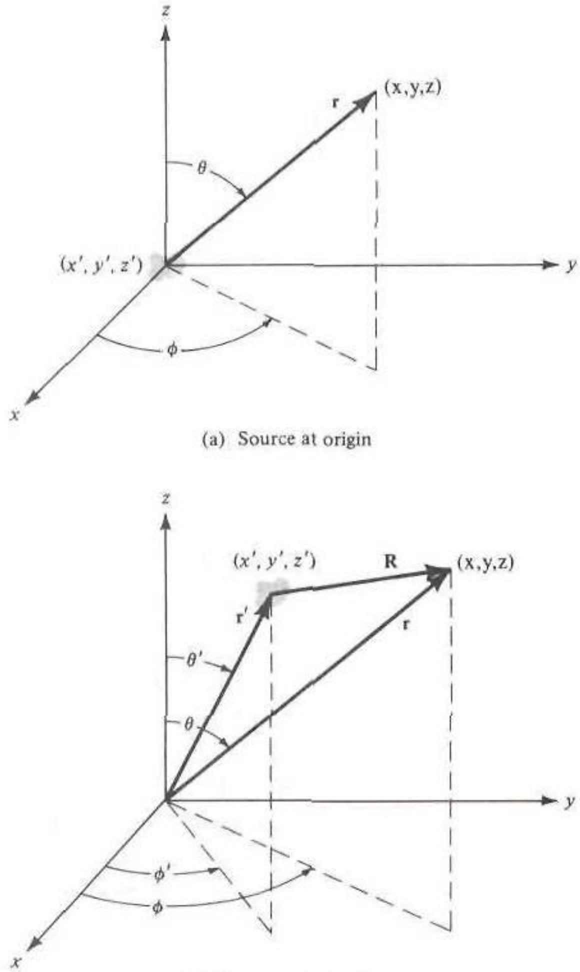

Figure 3.2 Coordinate systems for computing radiating fields.

which when expanded reduces to

$$\frac{d^2A_z(r)}{dr^2} + \frac{2}{r}\frac{dA_z(r)}{dr} + k^2A_z(r) = 0 \tag{3-34}$$

The partial derivative has been replaced by the ordinary derivative since A_z is only a function of the radial coordinate.

The differential equation of (3-34) has two independent solutions

$$A_{z1} = C_1\frac{e^{-jkr}}{r} \tag{3-35}$$

$$A_{z2} = C_2\frac{e^{+jkr}}{r} \tag{3-36}$$

Equation (3-35) represents an outwardly (in the radial direction) traveling wave and (3-36) describes an inwardly traveling wave (assuming an $e^{j\omega t}$ time variation). For this problem, the source is placed at the origin with the radiated fields traveling in the outward radial direction. Therefore, we choose the solution of (3-35), or

$$A_z = A_{z1} = C_1 \frac{e^{-jkr}}{r} \tag{3-37}$$

In the static case ($\omega = 0$, $k = 0$), (3-37) simplifies to

$$A_z = \frac{C_1}{r} \tag{3-38}$$

which is a solution to the wave equation of (3-32), (3-33), or (3-34) when $k = 0$. Thus at points removed from the source, the time-varying and the static solutions of (3-37) and (3-38) differ only by the e^{-jkr} factor; or the time-varying solution of (3-37) can be obtained by multiplying the static solution of (3-38) by e^{-jkr}.

In the presence of the source ($J_z \neq 0$) and $k = 0$ the wave equation of (3-31) reduces to

$$\nabla^2 A_z = -\mu J_z \tag{3-39}$$

This equation is recognized to be Poisson's equation whose solution is widely documented. The most familiar equation with Poisson's form is that relating the scalar electric potential ϕ to the electric charge density ρ. This is given by

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} \tag{3-40}$$

whose solution is

$$\phi = \frac{1}{4\pi\epsilon} \iiint_V \frac{\rho}{r}\, dv' \tag{3-41}$$

where r is the distance from any point on the charge density to the observation point. Since (3-39) is similar in form to (3-40), its solution is similar to (3-41), or

$$A_z = \frac{\mu}{4\pi} \iiint_V \frac{J_z}{r}\, dv' \tag{3-42}$$

Equation (3-42) represents the solution to (3-31) when $k = 0$ (static case). Using the comparative analogy between (3-37) and (3-38), the time-varying solution of (3-31) can be obtained by multiplying the static solution of (3-42) by e^{-jkr}. Thus

$$\boxed{A_z = \frac{\mu}{4\pi} \iiint_V J_z \frac{e^{-jkr}}{r}\, dv'} \tag{3-43}$$

which is a solution to (3-31).

If the current densities were in the x- and y-directions (J_x and J_y), the wave equation for each would reduce to

$$\nabla^2 A_x + k^2 A_x = -\mu J_x \tag{3-44}$$

$$\nabla^2 A_y + k^2 A_y = -\mu J_y \tag{3-45}$$

with corresponding solutions similar in form to (3-43), or

$$\boxed{A_x = \frac{\mu}{4\pi}\iiint_V J_x \frac{e^{-jkr}}{r}\,dv'} \tag{3-46}$$

$$\boxed{A_y = \frac{\mu}{4\pi}\iiint_V J_y \frac{e^{-jkr}}{r}\,dv'} \tag{3-47}$$

The solutions of (3-43), (3-46), and (3-47) allow us to write the solution to the vector wave equation of (3-14) as

$$\boxed{\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J} \frac{e^{-jkr}}{r}\,dv'} \tag{3-48}$$

If the source is removed from the origin and placed at a position represented by the primed coordinates (x', y', z'), as shown in Figure 3.2(b), (3-48) can be written as

$$\boxed{\mathbf{A}(x, y, z) = \frac{\mu}{4\pi}\iiint_V \mathbf{J}(x', y', z') \frac{e^{-jkR}}{R}\,dv'} \tag{3-49}$$

where the primed coordinates represent the source, the unprimed the observation point, and R the distance from any point on the source to the observation point. In a similar fashion we can show that the solution of (3-25) is given by

$$\boxed{\mathbf{F}(x, y, z) = \frac{\epsilon}{4\pi}\iiint_V \mathbf{M}(x', y', z') \frac{e^{-jkR}}{R}\,dv'} \tag{3-50}$$

If $\mathbf{J}$ and $\mathbf{M}$ represent linear densities (m^{-1}), (3-49) and (3-50) reduce to surface integrals, or

$$\boxed{\mathbf{A} = \frac{\mu}{4\pi}\iint_S \mathbf{J}_s(x', y', z') \frac{e^{-jkR}}{R}\,ds'} \tag{3-51}$$

$$\boxed{\mathbf{F} = \frac{\epsilon}{4\pi}\iint_S \mathbf{M}_s(x', y', z') \frac{e^{-jkR}}{R}\,ds'} \tag{3-52}$$

For electric and magnetic currents $\mathbf{I}_e$ and $\mathbf{I}_m$, they in turn reduce to line integrals of the form

$$\boxed{\mathbf{A} = \frac{\mu}{4\pi}\int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R}\,dl'} \tag{3-53}$$

$$\boxed{\mathbf{F} = \frac{\epsilon}{4\pi}\int_C \mathbf{I}_m(x', y', z')\frac{e^{-jkR}}{R}\,dl'} \qquad (3\text{-}54)$$

3.6 FAR-FIELD RADIATION

The fields radiated by antennas of finite dimensions are spherical waves. For these radiators, a general solution to the vector wave equation of (3-14) in spherical components, each as a function of r, θ, ϕ, takes the general form of

$$\mathbf{A} = \hat{\mathbf{a}}_r A_r(r, \theta, \phi) + \hat{\mathbf{a}}_\theta A_\theta(r, \theta, \phi) + \hat{\mathbf{a}}_\phi A_\phi(r, \theta, \phi) \qquad (3\text{-}55)$$

The amplitude variations of r in each component of (3-55) are of the form $1/r^n$, $n = 1, 2, \ldots$ [1]. Neglecting higher order terms of $1/r^n$ ($1/r^n = 0$, $n = 2, 3, \ldots$) reduces (3-55) to

$$\mathbf{A} \simeq [\hat{\mathbf{a}}_r A'_r(\theta, \phi) + \hat{\mathbf{a}}_\theta A'_\theta(\theta, \phi) + \hat{\mathbf{a}}_\phi A'_\phi(\theta, \phi)]\,\frac{e^{-jkr}}{r}, \qquad r \to \infty \qquad (3\text{-}56)$$

The r variations are separable from those of θ and ϕ. This will be demonstrated in the chapters that follow by many examples.

Substituting (3-56) into (3-15) reduces it to

$$\mathbf{E} = \frac{1}{r}\{-j\omega e^{-jkr}[\hat{\mathbf{a}}_r(0) + \hat{\mathbf{a}}_\theta A'_\theta(\theta, \phi) + \hat{\mathbf{a}}_\phi A'_\phi(\theta, \phi)]\} + \frac{1}{r^2}\{\cdots\} + \cdots \qquad (3\text{-}57)$$

The radial $\mathbf{E}$-field component has no $1/r$ terms, because its contributions from the first and second terms of (3-15) cancel each other.

Similarly, by using (3-56), we can write (3-2a) as

$$\mathbf{H} = \frac{1}{r}\left\{j\frac{\omega}{\eta}e^{-jkr}[\hat{\mathbf{a}}_r(0) + \hat{\mathbf{a}}_\theta A'_\phi(\theta, \phi) - \hat{\mathbf{a}}_\phi A'_\theta(\theta, \phi)]\right\} + \frac{1}{r^2}\{\cdots\} + \cdots \qquad (3\text{-}57a)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the medium.

Neglecting higher order terms of $1/r^n$, the radiated $\mathbf{E}$- and $\mathbf{H}$-fields have only θ and ϕ components. They can be expressed as

Far-Field Region

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \Rightarrow \boxed{\mathbf{E}_A \simeq -j\omega\mathbf{A}} \qquad (3\text{-}58a)$$

(for the θ and ϕ components only since $E_r \simeq 0$)

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq +j\frac{\omega}{\eta}A_\phi = -\frac{E_\phi}{\eta} \\ H_\phi &\simeq -j\frac{\omega}{\eta}A_\theta = +\frac{E_\theta}{\eta} \end{aligned}\right\} \Rightarrow \boxed{\mathbf{H}_A \simeq \frac{\hat{\mathbf{a}}_r}{\eta}\times\mathbf{E}_A = -j\frac{\omega}{\eta}\,\hat{\mathbf{a}}_r\times\mathbf{A}} \qquad (3\text{-}58b)$$

(for the θ and ϕ components only since $H_r \simeq 0$)

Radial field components exist only for higher order terms of $1/r^n$.

In a similar manner, the far-zone fields due to a magnetic source **M** (potential **F**) can be written as

Far-Field Region

$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq -j\omega F_\theta \\ H_\phi &\simeq -j\omega F_\phi \end{aligned}\right\} \Rightarrow \boxed{\mathbf{H}_F \simeq -j\omega\,\mathbf{F}} \tag{3-59a}$$

(for the θ and ϕ components only since $H_r \simeq 0$)

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega\eta F_\phi = \eta H_\phi \\ E_\phi &\simeq +j\omega\eta F_\theta = -\eta H_\theta \end{aligned}\right\} \Rightarrow \boxed{\mathbf{E}_F = -\eta\hat{\mathbf{a}}_r \times \mathbf{H}_F = j\omega\eta\hat{\mathbf{a}}_r \times \mathbf{F}} \tag{3-59b}$$

(for the θ and ϕ components only since $E_r \simeq 0$)

Simply stated, *the corresponding far-zone* **E**- *and* **H**-*field components are orthogonal to each other and form TEM (to r) mode fields.* This is a very useful relation, and it will be adopted in the chapters that follow for the solution of the far-zone radiated fields. The far-zone (far-field) region for a radiator is defined in Figure 2.5. Its smallest radial distance is $2D^2/\lambda$ where D is the largest dimension of the radiator.

3.7 DUALITY THEOREM

When two equations that describe the behavior of two different variables are of the same mathematical form, their solutions will also be identical. The variables in the two equations that occupy identical positions are known as *dual* quantities and a solution of one can be formed by a systematic interchange of symbols to the other. This concept is known as the *duality theorem.*

Comparing Equations (3-2a), (3-3), (3-10), (3-14), and (3-15) to (3-16), (3-17), (3-21), (3-25), and (3-26), respectively, it is evident that they are to each other dual equations and their variables dual quantities. Thus knowing the solutions to one set (i.e., $\mathbf{J} \neq 0, \mathbf{M} = 0$), the solution to the other set ($\mathbf{J} = 0, \mathbf{M} \neq 0$) can be formed by a proper interchange of quantities. The dual equations and their dual quantities are listed in Tables 3.1 and 3.2 for electric and magnetic sources, respectively. Duality

Table 3.1 DUAL EQUATIONS FOR ELECTRIC (**J**) AND MAGNETIC (**M**) CURRENT SOURCES

Electric Sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic Sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A$	$\nabla \times \mathbf{H}_F = j\omega\epsilon\mathbf{E}_F$
$\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\epsilon\mathbf{E}_A$	$-\nabla \times \mathbf{E}_F = \mathbf{M} + j\omega\mu\mathbf{H}_F$
$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}$	$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M}$
$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkR}}{R}\,dv'$	$\mathbf{F} = \frac{\epsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-jkR}}{R}\,dv'$
$\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}$	$\mathbf{E}_F = -\frac{1}{\epsilon}\nabla \times \mathbf{F}$
$\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{A})$	$\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F})$

Table 3.2 DUAL QUANTITIES FOR ELECTRIC (**J**) AND MAGNETIC (**M**) CURRENT SOURCES

Electric Sources ($\mathbf{J} \neq 0$, $\mathbf{M} = 0$)	Magnetic Sources ($\mathbf{J} = 0$, $\mathbf{M} \neq 0$)
$\mathbf{E}_A$	$\mathbf{H}_F$
$\mathbf{H}_A$	$-\mathbf{E}_F$
$\mathbf{J}$	$\mathbf{M}$
$\mathbf{A}$	$\mathbf{F}$
ϵ	μ
μ	ϵ
k	k
η	$1/\eta$
$1/\eta$	η

only serves as a guide to form mathematical solutions. It can be used in an abstract manner to explain the motion of magnetic charges giving rise to magnetic currents, when compared to their dual quantities of moving electric charges creating electric currents. It must, however, be emphasized that this is purely mathematical in nature since it is known as of today, that there are no magnetic charges or currents in nature.

3.8 RECIPROCITY AND REACTION THEOREMS

We are all well familiar with the reciprocity theorem, as applied to circuits, which states that "in any network composed of linear, bilateral, lumped elements, if one places a constant **current** *(voltage)* generator between two **nodes** *(in any branch)* and places a **voltage** *(current)* meter between any other two **nodes** *(in any other branch)*, makes observation of the meter reading, then interchanges the locations of the source and the meter, the meter reading will be unchanged" [2]. We want now to discuss the reciprocity theorem as it applies to electromagnetic theory. This is done best by the use of Maxwell's equations.

Let us assume that within a linear and isotropic medium, but not necessarily homogeneous, there exist two sets of sources $\mathbf{J}_1$, $\mathbf{M}_1$, and $\mathbf{J}_2$, $\mathbf{M}_2$ which are allowed to radiate simultaneously or individually inside the same medium at the frequency and produce fields $\mathbf{E}_1$, $\mathbf{H}_1$ and $\mathbf{E}_2$, $\mathbf{H}_2$, respectively. It can be shown [1], [3] that the sources and fields satisfy

$$-\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2 \qquad (3\text{-}60)$$

which is called the *Lorentz Reciprocity Theorem* in differential form.

Taking a volume integral of both sides of (3-60) and using the divergence theorem on the left side, we can write it as

$$-\oiint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s}'$$
$$= \iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv' \qquad (3\text{-}61)$$

which is designated as the *Lorentz Reciprocity Theorem* in integral form.

For a source-free ($\mathbf{J}_1 = \mathbf{J}_2 = \mathbf{M}_1 = \mathbf{M}_2 = 0$) region, (3-60) and (3-61) reduce, respectively, to

$$\boxed{\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0} \tag{3-62}$$

and

$$\boxed{\oiint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s}' = 0} \tag{3-63}$$

Equations (3-62) and (3-63) are special cases of the Lorentz Reciprocity Theorem and must be satisfied in source-free regions.

As an example of where (3-62) and (3-63) may be applied and what they would represent, consider a section of a waveguide where two different modes exist with fields $\mathbf{E}_1$, $\mathbf{H}_1$ and $\mathbf{E}_2$, $\mathbf{H}_2$. For the expressions of the fields for the two modes to be valid, they must satisfy (3-62) and/or (3-63).

Another useful form of (3-61) is to consider that the fields ($\mathbf{E}_1$, $\mathbf{H}_1$, $\mathbf{E}_2$, $\mathbf{H}_2$) and the sources ($\mathbf{J}_1$, $\mathbf{M}_1$, $\mathbf{J}_2$, $\mathbf{M}_2$) are within a medium that is enclosed by a sphere of infinite radius. Assume that the sources are positioned within a finite region and that the fields are observed in the far field (ideally at infinity). Then the left side of (3-61) is equal to zero, or

$$\oiint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{s}' = 0 \tag{3-64}$$

which reduces (3-61) to

$$\iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv' = 0 \tag{3-65}$$

Equation 3-65 can also be written as

$$\boxed{\iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv' = \iiint_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1)\, dv'} \tag{3-66}$$

The reciprocity theorem, as expressed by (3-66), is the most useful form.

A close observation of (3-61) will reveal that it does not, in general, represent relations of power because no conjugates appear. The same is true for the special cases represented by (3-63) and (3-66). Each of the integrals in (3-66) can be interpreted as a coupling between a set of fields and a set of sources, which produce another set of fields. This coupling has been defined as *Reaction* [4] and each of the integrals in (3-66) are denoted by

$$\langle 1, 2 \rangle = \iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv \tag{3-67}$$

$$\langle 2, 1 \rangle = \iiint_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1)\, dv \tag{3-68}$$

The relation $\langle 1, 2\rangle$ of (3-67) relates the reaction (coupling) of fields ($\mathbf{E}_1$, $\mathbf{H}_1$), which are produced by sources $\mathbf{J}_1$, $\mathbf{M}_1$ to sources ($\mathbf{J}_2$, $\mathbf{M}_2$), which produce fields $\mathbf{E}_2$, $\mathbf{H}_2$; $\langle 2, 1\rangle$ relates the reaction (coupling) of fields ($\mathbf{E}_2$, $\mathbf{H}_2$) to sources ($\mathbf{J}_1$, $\mathbf{M}_1$). For reciprocity to hold, it requires that the reaction (coupling) of one set of sources with the corresponding fields of another set of sources must be equal to the reaction (coupling) of the second set of sources with the corresponding fields of the first set of sources, and vice versa. In equation form, it is written as

$$\langle 1, 2\rangle = \langle 2, 1\rangle \tag{3-69}$$

3.8.1 Reciprocity for Two Antennas

There are many applications of the reciprocity theorem. To demonstrate its potential, an antenna example will be considered. Two antennas, whose input impedances are Z_1 and Z_2, are separated by a linear and isotropic (but not necessarily homogeneous) medium, as shown in Figure 3.3. One antenna (#1) is used as a transmitter and the other (#2) as a receiver. The equivalent network of each antenna is given in Figure 3.4. The internal impedance of the generator Z_g is assumed to be the conjugate of the impedance of antenna #1 ($Z_g = Z_1^* = R_1 - jX_1$) while the load impedance Z_L is equal to the conjugate of the impedance of antenna #2 ($Z_L = Z_2^* = R_2 - jX_2$). These assumptions are made only for convenience.

The power delivered by the generator to antenna #1 is given by

$$P_1 = \frac{1}{2}\,\text{Re}[V_1 I_1^*] = \frac{1}{2}\,\text{Re}\left[\left(\frac{V_g Z_1}{Z_1 + Z_g}\right)\frac{V_g^*}{(Z_1 + Z_g)^*}\right] = \frac{|V_g|^2}{8R_1} \tag{3-70}$$

If the transfer admittance of the combined network consisting of the generator impedance, antennas, and load impedance is Y_{21}, the current through the load is $V_g Y_{21}$ and the power delivered to the load is

$$P_2 = \tfrac{1}{2}\,\text{Re}[Z_2(V_g Y_{21})(V_g Y_{21})^*] = \tfrac{1}{2}R_2|V_g|^2|Y_{21}|^2 \tag{3-71}$$

The ratio of (3-69) to (3-68) is

$$\frac{P_2}{P_1} = 4R_1R_2|Y_{21}|^2 \tag{3-72}$$

In a similar manner, we can show that when antenna #2 is transmitting and #1 is receiving, the power ratio of P_1/P_2 is given by

$$\frac{P_1}{P_2} = 4R_2R_1|Y_{12}|^2 \tag{3-73}$$

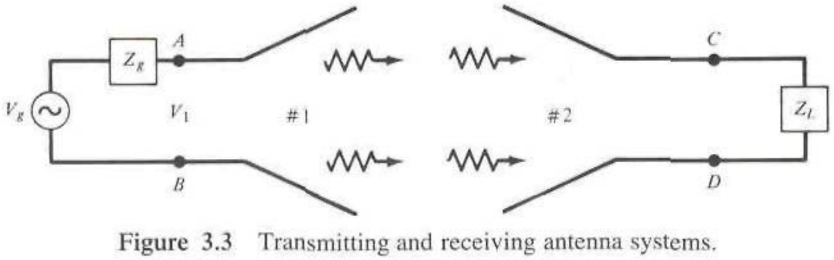

Figure 3.3 Transmitting and receiving antenna systems.

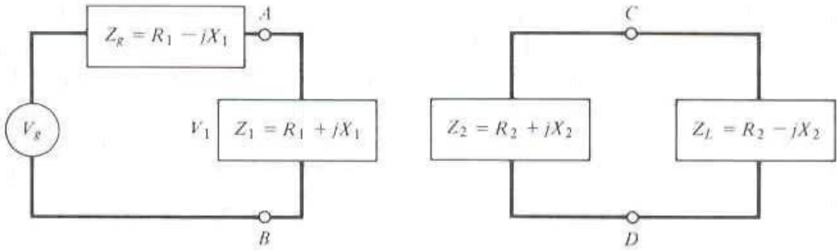

Figure 3.4 Two antenna systems with conjugate loads.

Under conditions of reciprocity ($Y_{12} = Y_{21}$), the power delivered in either direction is the same.

3.8.2 Reciprocity for Radiation Patterns

The radiation pattern is a very important antenna characteristic. Although it is usually most convenient and practical to measure the pattern in the receiving mode, it is identical, because of reciprocity, to that of the transmitting mode.

Reciprocity for antenna patterns is general provided the materials used for the antennas and feeds, and the media of wave propagation are linear. Nonlinear devices, such as diodes, can make the antenna system nonreciprocal. The antennas can be of any shape or size, and they do not have to be matched to their corresponding feed lines or loads provided there is a distinct single propagating mode at each port. The only other restriction for reciprocity to hold is for the antennas in the transmit and receive modes are polarization matched, including the sense of rotation. This is necessary so that the antennas can transmit and receive the same field components, and thus total power. If the antenna that is used as a probe to measure the fields radiated by the antenna under test is not of the same polarization, then in some situations the transmit and receive patterns can still be the same. For example, if the transmit antenna is circularly polarized and the probe antenna is linearly polarized, then if the linearly polarized probe antenna is used twice and it is oriented one time to measure the θ-component and the other the ϕ-component, then the sum of the two components can represent the pattern of the circularly polarized antenna in either the transmit or receive modes. During this procedure, the power level and sensitivities must be held constant.

To detail the procedure and foundation of pattern measurements and reciprocity, let us refer to Figures 3.5(a) and (b). The antenna under test is #1 while the probe antenna (#2) is oriented to transmit or receive maximum radiation. The voltages and currents V_1, I_1 at terminals 1–1 of antenna #1 and V_2, I_2 at terminals 2–2 of antenna #2 are related by

$$\begin{aligned} V_1 &= Z_{11}I_1 + Z_{12}I_2 \\ V_2 &= Z_{21}I_1 + Z_{22}I_2 \end{aligned} \tag{3-74}$$

where

Z_{11} = self-impedance of antenna #1
Z_{22} = self-impedance of antenna #2
Z_{12}, Z_{21} = mutual impedances between antennas #1 and #2

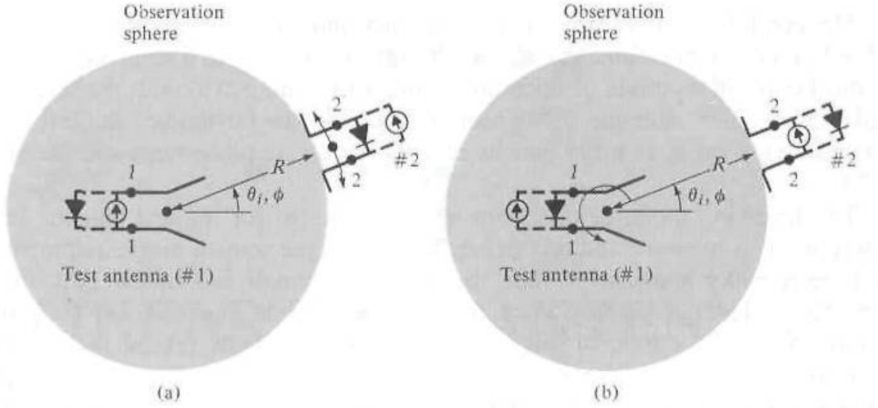

Figure 3.5 Antenna arrangement for pattern measurements and reciprocity theorem.

If a current I_1 is applied at the terminals 1–1 and voltage V_2 (designated as V_{2oc}) is measured at the *open* ($I_2 = 0$) terminals of antenna #2, then an equal voltage V_{1oc} will be measured at the *open* ($I_1 = 0$) terminals of antenna #1 provided the current I_2 of antenna #2 is equal to I_1. In equation form, we can write

$$Z_{21} = \left.\frac{V_{2oc}}{I_1}\right|_{I_2=0} \tag{3-75a}$$

$$Z_{12} = \left.\frac{V_{1oc}}{I_2}\right|_{I_1=0} \tag{3-75b}$$

If the medium between the two antennas is linear, passive, isotropic, and the waves monochromatic, then because of reciprocity

$$Z_{21} = \left.\frac{V_{2oc}}{I_1}\right|_{I_2=0} = \left.\frac{V_{1oc}}{I_2}\right|_{I_1=0} = Z_{12} \tag{3-76}$$

If in addition $I_1 = I_2$, then

$$V_{2oc} = V_{1oc} \tag{3-77}$$

The above are valid for any position and any mode of operation between the two antennas.

Reciprocity will now be reviewed for two modes of operation. In one mode, antenna #1 is held stationary while #2 is allowed to move on the surface of a constant radius sphere, as shown in Figure 3.5(a). In the other mode, antenna #2 is maintained stationary while #1 pivots about a point, as shown in Figure 3.5(b).

In the mode of Figure 3.5(a), antenna #1 can be used either as a transmitter or receiver. In the transmitting mode, while antenna #2 is moving on the constant radius sphere surface, the open terminal voltage V_{2oc} is measured. In the receiving mode, the open terminal voltage V_{1oc} is recorded. The three-dimensional plots of V_{2oc} and V_{1oc}, as a function of θ and ϕ, have been defined in Section 2.2 as *field patterns*. Since the three-dimensional graph of V_{2oc} is identical to that of V_{1oc} (due to reciprocity), the *transmitting* (V_{2oc}) *and receiving* (V_{1oc}) field patterns are also equal. The same conclusion can be arrived at if antenna #2 is allowed to remain stationary while #1 rotates, as shown in Figure 3.5(b).

The conditions of reciprocity hold whether antenna #1 is used as a transmitter and #2 as a receiver *or* antenna #2 as a transmitter and #1 as a receiver. In practice, the most convenient mode of operation is that of Figure 3.5(b) with the test antenna used as a receiver. Antenna #2 is usually placed in the far-field of the test antenna (#1), and vice-versa, in order that its radiated fields are plane waves in the vicinity of #1.

The receiving mode of operation of Figure 3.5(b) for the test antenna is most widely used to measure antenna patterns, because the transmitting equipment is in most cases bulky and heavy while the receiver is small and lightweight. In some cases, the receiver is nothing more than a simple diode detector. The transmitting equipment usually consists of sources and amplifiers. To make precise measurements, especially at microwave frequencies, it is necessary to have frequency and power stabilities. Therefore, the equipment must be placed on stable and vibration-free platforms. This can best be accomplished by allowing the transmitting equipment to be held stationary and the receiving equipment to rotate.

An excellent manuscript on test procedures for antenna measurements of amplitude, phase, impedance, polarization, gain, directivity, efficiency, and others has been published by IEEE [5]. A condensed summary of it is found in [6], and a review is presented in Chapter 15 of this text.

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PROBLEMS

3.1. If $\mathbf{H}_e = j\omega\epsilon\nabla \times \mathbf{\Pi}_e$, where $\mathbf{\Pi}_e$ is the electric Hertzian potential, show that

(a) $\nabla^2\mathbf{\Pi}_e + k^2\mathbf{\Pi}_e = j\dfrac{1}{\omega\epsilon}\mathbf{J}$ (b) $\mathbf{E}_e = k^2\mathbf{\Pi}_e + \nabla(\nabla\cdot\mathbf{\Pi}_e)$

(c) $\mathbf{\Pi}_e = -j\dfrac{1}{\omega\mu\epsilon}\mathbf{A}$

3.2. If $\mathbf{E}_h = -j\omega\mu\nabla \times \mathbf{\Pi}_h$, where $\mathbf{\Pi}_h$ is the magnetic Hertzian potential, show that

(a) $\nabla^2\mathbf{\Pi}_h + k^2\mathbf{\Pi}_h = j\dfrac{1}{\omega\mu}\mathbf{M}$ (b) $\mathbf{H}_h = k^2\mathbf{\Pi}_h + \nabla(\nabla\cdot\mathbf{\Pi}_h)$

(c) $\mathbf{\Pi}_h = -j\dfrac{1}{\omega\mu\epsilon}\mathbf{F}$

3.3. Verify that (3-35) and (3-36) are solutions to (3-34).

3.4. Show that (3-42) is a solution to (3-39) and (3-43) is a solution to (3-31).

3.5. Verify (3-57) and (3-57a).

3.6. Derive (3-60) and (3-61).